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Mh4718 Week 10

Week 10

0.1 Solving Differential Equations (contd.)

0.1.1 Separation of variables

The technique of separation of variables uses the so-called *substitution rule* for integration. The *substitution rule* is in turn based on the *chain rule* for differentiation.

Recall that, according to the chain rule (assuming that all functions are suitably differentiable), we have

$$\frac{d}{dx} f(u(x)) = \frac{d}{du} f(u) \frac{d}{dx} u(x)$$

Example 0.1

$$\frac{d}{dx} \sin(x^2) = 2x \cos(x^2)$$

Recall also that an indefinite integral is an *anti-derivative* i.e.

$$\int \left(\frac{d}{dx} f(x) \right) \mathrm{d}x = f(x).$$

Example 0.2

$$\int x^2 \mathrm{d}x = \tfrac{1}{3}x^3 + constant$$

because

$$\frac{d}{dx}(\tfrac{1}{3}x^3 + constant) = x^2.$$

Therefore we can see that

$$\int \left(\frac{d}{du} f(u) \frac{d}{dx} u(x) \right) \mathrm{d}x = f(u(x)).$$

Example 0.3

$$\int 2x \cos(x^2) \mathrm{d}x = \sin(x).$$

The *substitution rule* is now obvious because

$$\int \left(\frac{d}{du} f(u) \frac{d}{dx} u(x) \right) \mathrm{d}x = f(u(x)) = \int \frac{d}{du} f(u) \mathrm{d}u.$$

Notationally we see that

$$\frac{d}{dx} u(x) \mathrm{d}x$$

in the left hand integral has been replaced by

$$\mathrm{d}u$$

in the right hand integral (as if $\mathrm{d}x$ has been cancelled!)
Thus we get the *substitution rule*

$$\int F(u) \frac{du}{dx} \mathrm{d}x \to \int F(u) \mathrm{d}u.$$

Example 0.4

$$\int \overset{\frac{du}{dx}}{\overset{\downarrow}{2x}} \cos(\overset{u}{\overset{\downarrow}{x^2}}) \mathrm{d}x = \int \cos(u) \mathrm{d}u = \sin(u) = \sin(x^2)$$

Now if, in an IVP

$$\frac{dy}{dx} = F(x,y), y(x_0) = y_0,$$

we have

$$F(x,y) = a(x)b(y)$$

(i.e. the variables can be separated) then we have

$$\frac{dy}{dx} = a(x)b(y) \Rightarrow \frac{1}{b(y)}\frac{dy}{dx} = a(x) \text{ provided } b(y_0) \neq 0$$

Then

$$\int \frac{1}{b(y)}\frac{dy}{dx}\mathrm{d}x = \int a(x)\mathrm{d}x$$

and we can see that

$$\int \frac{1}{b(y)}\frac{dy}{dx}\mathrm{d}x = \int \frac{1}{b(y)}\mathrm{d}y.$$

And we have

$$\int \frac{1}{b(y)}\mathrm{d}y = \int a(x)\mathrm{d}x$$

Example 0.5

(i) Solve the IVP $\frac{dy}{dx} = \frac{3y-3}{x}, y(1) = 2$ by separation of variables.

$$\begin{aligned}
\frac{dy}{dx} = \frac{3y-3}{x} &\Rightarrow \frac{1}{3y-3}\frac{dy}{dx} = \frac{1}{x} \\
&\Rightarrow \int \frac{1}{3y-3}\frac{dy}{dx}\mathrm{d}x = \int \frac{1}{x}\mathrm{d}x \\
&\Rightarrow \int \frac{1}{3y-3}\mathrm{d}y = \int \frac{1}{x}\mathrm{d}x \\
&\Rightarrow \tfrac{1}{3}\ln(y-1) = \ln(x) + C_1, \quad C_1 \in \mathbb{R} \\
&\Rightarrow \ln(y-1) = 3\ln(x) + C = \ln(x^3) + C \\
&\Rightarrow y - 1 = e^{\ln(x^3)+C} = e^{\ln(x^3)}e^C = Ke^{\ln(x^3)}, K \in \mathbb{R} \\
&\Rightarrow y = Ke^{\ln(x^3)} + 1 = Kx^3 + 1
\end{aligned}$$

The initial values are $y(1) = 2$ therefore

$$K + 1 = 2 \Rightarrow K = 1.$$

And so the solution to the IVP is $y = x^3 + 1$.

(ii) Solve the IVP $\frac{dy}{dx} = y\cos(x), y(0) = 1$ by separation of variables.

$$\begin{aligned}\frac{dy}{dx} = y\cos(x) &\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cos(x)\\ &\Rightarrow \int \frac{1}{y}\frac{dy}{dx}\mathrm{d}x = \int \cos(x)\mathrm{d}x\\ &\Rightarrow \int \frac{1}{y}\mathrm{d}y = \int \cos(x)\mathrm{d}x\\ &\Rightarrow \ln(y) = \sin(x) + C, C \in \mathbb{R}\\ &\Rightarrow y = e^{\sin(x)+C} = Ke^{\sin(x)}\end{aligned}$$

The initial values are $y(0) = 1$ therefore $K = 1$.
And so the solution to the IVP is $y = e^{\sin(x)}$.

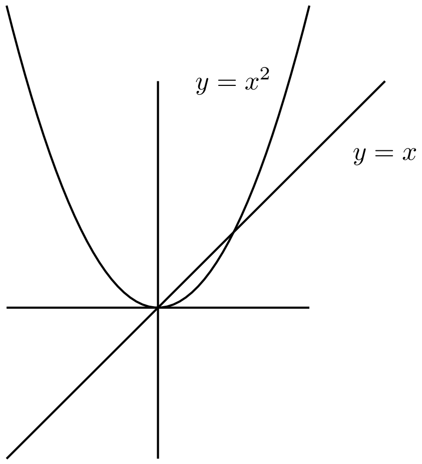
0.2 Fixed point iteration.

Let F be a real valued function whose domain is a subset of $\mathbb{R}$. A point $p \in \mathbb{R}$ is said to be a *fixed point* of F if $F(p) = p$.

Example 0.6

Let $F(x) = x^2$. We see that $F(0) = 0$ and $F(1) = 1$ and so 0 and 1 are fixed points of F.

If p is a fixed point of F then $(p, F(p)) = (p, p)$ which means that the point (p, p) will be on the graph of F and on the straight line $y = x$.
So we can get an idea whether a function F has a fixed point or not by sketching its graph and noting whether it intersects the line $y = x$ or not.



In certain favourable circumstances (to be discussed) the following iteration method leads to a fixed point of a given function $F(x)$. That is, we discover a real number p such that $F(p) = p$:

First choose x_0 "close enough" to p (by sketching a graph for instance) then define:

$$x_{n+1} = F(x_n), n = 0, 1, 2 \dots$$

If we let $x_1 = F(x_0), x_2 = F(x_1), x_3 = F(x_3)$ etc. then in certain circumstances (to be discussed) the sequence

$$x_0, x_1, x_2, x_3 \dots$$

will converge to p a fixed point of $F(x)$.

Example 0.7

(i) Use Excel to estimate a solution for $x = 1 + 0.5\sin(x)$ by fixed point iteration.
First graph $y = x = 1 + 0.5\sin(x)$ and $y = x$ on the same axes using a speculative domain until you the two intersect somewhere. Use this chart to estimate x_0.

(ii) Use Excel to estimate a solution for $3 + 2\sin(x)$ by fixed point iteration.
First graph $3 + 2\sin(x)$ and $y = x$ on the same axes using a speculative domain until you the two intersect somewhere. Use this chart to estimate x_0.

The following is a basic C++ routine for fixed point iteration:

```
#include <iostream>
#include <cmath>
#include <iomanip>


using namespace std;
double F(double x)
{
    return *******;  //place the formula for the function here
}
void main()
{
    double x=1; //Initial value for x
    double nx,distance;
    cout<<setprecision(20);
    do
    {
        nx=F(x);
        distance =fabs(nx-x);
        x=nx;
    }while(distance >0.0000001); //adjust this figure for more or less accuracy
    cout<<"The limit is approximately "<<x<<endl;

}
```